

Pursuing parameters for critical-density dark matter models

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5 February 2008

ABSTRACT

We present an extensive comparison of models of structure formation with observations, based on linear and quasi-linear theory. We assume a critical matter density, and study both cold dark matter models and cold plus hot dark matter models. We explore a wide range of parameters, by varying the fraction of hot dark matter Ω_ν , the Hubble parameter h and the spectral index of density perturbations n , and allowing for the possibility of gravitational waves from inflation influencing large-angle microwave background anisotropies. New calculations are made of the transfer functions describing the linear power spectrum, with special emphasis on improving the accuracy on short scales where there are strong constraints. For assessing early object formation, the transfer functions are explicitly evaluated at the appropriate redshift. The observations considered are the four-year *COBE* observations of microwave background anisotropies, peculiar velocity flows, the galaxy correlation function, and the abundances of galaxy clusters, quasars and damped Lyman alpha systems. Each observation is interpreted in terms of the power spectrum filtered by a top-hat window function. We find that there remains a viable region of parameter space for critical-density models when all the dark matter is cold, though h must be less than 0.5 before any fit is found and n significantly below unity is preferred. Once a hot dark matter component is invoked, a wide parameter space is acceptable, including $n \simeq 1$. The allowed region is characterized by $\Omega_\nu \lesssim 0.35$ and $0.60 \lesssim n \lesssim 1.25$, at 95 per cent confidence on at least one piece of data. There is no useful lower bound on h , and for curious combinations of the other parameters it is possible to fit the data with h as high as 0.65.

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Key words: cosmology: theory – dark matter.

1 INTRODUCTION

The concept of cosmological inflation has motivated an enormous amount of research into the formation of structure in the Universe. It has long been known that the simplest particle physics models for inflation typically predict that the Universe is spatially flat and that the gravitational seeds for structure are adiabatic, Gaussian-distributed density fluctuations with a nearly Harrison–Zel’dovich spectrum (power spectrum index of $n \sim 1$). In order to proceed with detailed

calculations from this starting point, one needs to pick a value for the Hubble constant. The standard choice has been to assume $h = 0.5$, where the present Hubble constant is parametrized as $H_0 \equiv 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$. Finally, the character of the dark matter must be decided. Big bang nucleosynthesis implies that the bulk of dark matter cannot be baryonic.

The choice for this dark matter that involves the fewest assumptions is relic neutrinos, as we know they exist and

expect that they fill the Universe. Neutrinos are referred to as hot dark matter (HDM) because they remain relativistic until the horizon size is comparable to large-scale structures. Unfortunately HDM does not give a satisfactory picture of structure formation, because galaxies form too late and the phase space of the haloes of small galaxies is not large enough to accommodate the required number of neutrinos. A more popular alternative for the dark matter is to assume the existence of a cold relic particle, known as cold dark matter (CDM). Typical candidate particles for CDM are axions and the lightest supersymmetric particles.

Initial studies of galaxy formation found that the galaxies in CDM models were too clustered when used with inflationary-type fluctuations (see e.g. Davis et al. 1985). This problem was surmounted by introducing the concept of *biasing*, in which the fluctuations in the galactic distribution are much larger or ‘biased’ compared with the underlying density field. This model (with the ingredients $\Omega_{\text{CDM}} = 0.95$, $\Omega_{\text{B}} = 0.05$, $h = 0.5$ and strongly biased scale-invariant adiabatic Gaussian fluctuations) proved to be quite successful at explaining many properties of galaxies and clusters, mostly on smaller scales. However, because the amplitude of density fluctuations needed to be reduced to account properly for galaxies, this also meant that there would be insufficient power for making much larger scale structures.

Standard CDM’s problems with large-scale structure had already been noticed in the 1980s. In particular, the observed spatial correlation of galactic clusters was much stronger than predicted by the model. In the meantime, it was noted (Holman, Lazarides & Shafi 1983; Shafi 1983; Mohapatra & Senjanovic 1983) that certain realistic particle physics grand unification models predict the simultaneous presence of cold and hot dark matter. Such a mixture was recognized (Shafi & Stecker 1984) to have the desirable properties of reduced small-scale power to make galaxies properly, while still having significant amounts of power on larger scales. [Models that mixed hot and warm dark matter were also studied (Bonometto & Valdarnini 1984; Fang, Li & Xiang 1984; Valdarnini & Bonometto 1985).] However, it was pointed out (Acchilli, Occhionero & Scaramella 1985; Bardeen, Bond & Efstathiou 1987; van Dalen & Schaefer 1992) that, if the mixture contained more HDM than CDM, the model would have difficulty forming galaxies early enough. Thus this model became known as the cold plus hot dark matter model (CHDM), to indicate that there should be more CDM than HDM.

As evidence for more large-scale power than expected in CDM models continued to accumulate during the 1980s the outlook for the CHDM model brightened. The number densities of cosmic structures and large cluster correlation lengths in CHDM models were shown to be in better agreement with observations (Occhionero & Scaramella 1989; van Dalen & Schaefer 1992; Holtzman & Primack 1993). In addition to showing that CHDM predictions of observed large-scale bulk flows agreed better with observations, two papers predicted the amplitude of the temperature fluctuations expected on large angular scales in the cosmic microwave background radiation (Schaefer, Shafi & Stecker 1989; Holtzman 1989) well before the launch of the *Cosmic Background Explorer* (COBE) satellite. The verification by COBE of these anisotropy predictions brought about a wave of intense interest in the CHDM model.

The recent rapid increase in the quality of the observations of large-scale structure and microwave background temperature fluctuations has led to a new precision in investigations of theoretical models. Until recently, it was standard practice to derive conclusions about models of structure formation within a fairly rigid subset of assumptions about the relevant parameters. Two of these parameters are the spectral index n , and a parameter r specifying the relative contribution of gravitational waves to the cosmic microwave background anisotropy (Liddle & Lyth 1992), and they have usually been fixed at the canonical values $n = 1$ and $r = 0$. However, within a given model of inflation their values are determined or at least constrained, and while many models do accurately give these canonical values there are other models which do not. It is therefore realistic to allow n and r to vary when confronting a model with the data, and we shall adopt that viewpoint in this paper. An observational determination of these parameters in the future will be a powerful constraint on models of inflation, and hence on the nature of the fundamental interactions at very high energy scales. (For recent discussions of the relation between models of inflation and the fundamental interactions, see e.g. Schaefer & Shafi 1994; Copeland et al. 1994; Dvali, Shafi & Schaefer 1994; Stewart 1995a,b; Banks et al. 1995; Randall, Soljačić & Guth 1995; Ross & Sarkar 1996.)

Another parameter which is often fixed is the Hubble constant, usually to the value 0.5. Its variation can be important: the amount of small-scale power is extremely sensitive to the value of the Hubble parameter, as the redshift of matter domination scales quadratically with h . The choice of baryon density also can have a modest impact, as we discuss shortly. Our intent here is to study the CHDM model realistically, by varying the parameters of inflation and h to see which are the most favourable values by testing the model against data.

The above set of parameters is by no means complete, even within the limited context of inflation. For instance, inflation says nothing about whether or not there might be a relic cosmological constant Λ contributing to the present-day spatial flatness, although it may be difficult to understand the magnitude of the residual Λ within the philosophical context of inflation. In keeping with the original spirit of inflation, we set $\Lambda = 0$ here. Recently it has also been emphasized that one can obtain genuinely open universes from inflation, albeit at present only with considerable tuning of parameters. Structure formation is apparently viable in these models (Ratra & Peebles 1994; Górski et al. 1995; Liddle et al. 1996), but we shall not consider them here. Our assumption, therefore, is that the Universe possesses a critical density of matter.

Further impetus has been delivered to the CHDM model by the observations of neutrino oscillations from the Sun, atmospheric cosmic ray cascades, and possibly by the Liquid Scintillator Neutrino Detector (LSND) experiment (Caldwell 1994; Athanassopoulos et al. 1995). These observations suggest that some of the neutrino masses are non-zero, and that one or more neutrino species may provide a significant HDM density. It has even been suggested that a multiple (2 or 3) neutrino CHDM scenario (Shafi & Stecker 1984) may provide an even better fit to observational data (Primack et al. 1995; Pogosyan & Starobinsky 1995b; Babu, Schaefer & Shafi 1996). While promising, this remains specula-

tive physics. Here we will only consider the situation of a single few eV mass neutrino. Indeed such a scenario can be made reasonably compatible with all of the oscillation experimental results (see Babu et al. 1996 and references therein). We are also, of course, assuming the standard cosmology for the neutrinos (for some alternative proposals, see Kaiser, Malaney & Starkman (1993); Bonometto, Caldara & Masiero (1994); Pierpaoli & Bonometto (1995); Pierpaoli et al. (1996)).

While detailed N -body simulations, necessitating the selection of particular parameter values, are required to provide a detailed comparison of models against observations, it is vital to carry out an investigation of the wider parameter space using the less intensive strategy of linear perturbation theory in order to find those regions of parameter space best suited to matching the data. Linear theory and quasi-linear theory offer powerful tools for investigating the shape of the density perturbation power spectrum across a very wide range of scales, as there are now copious data addressing scales large enough to still be linear today. Further, even the shorter scales that are non-linear today can be investigated by examining phenomena such as the abundance of quasars and damped Lyman alpha systems at moderate redshift, corresponding to times when those scales were still in the linear regime.

A choice is required for the baryon density, which is taken to agree with standard nucleosynthesis. The theory of nucleosynthesis has seen some developments recently, and the range advocated by Walker et al. (1991) is now seen as too stringent, especially their upper limit. We choose to take a value compatible with more recent analyses by Copi, Schramm & Turner (1995a,b) and Hata et al. (1995) which is $\Omega_B h^2 = 0.016$. Copi et al. (1995a,b) claim that a plausible range of Ω_B , clearly intended to be thought of as 95 per cent confidence, extends to 50 per cent in either direction around that value. The slightly higher value is helpful, especially in models without a hot component, as it helps to remove short-scale power from the spectrum. In CDM models, there may be further motivation to raise it further towards the top end of the range (e.g. White et al. 1995b); in such models it is easy to quantify the benefit of raising Ω_B and we shall show how to do this later.

For our investigation, we shall therefore treat as our three main parameters Ω_ν , n and h , and in addition allow the incorporation of a gravitational wave component though the space of that parameter will not be as extensively explored. Early investigations typically only varied Ω_ν , but were followed by treatments by Schaefer & Shafi (1993), Liddle & Lyth (1993b) and Schaefer & Shafi (1994) who investigated the Ω_ν - n plane, both with and without gravitational waves but concentrating only on $n < 1$. Pogosyan & Starobinsky (1993) carried out an analogous investigation of the Ω_ν - h plane, fixing $n = 1$. More recently, Pogosyan & Starobinsky (1995a) made a study of the full Ω_ν - n - h parameter space, concluding that $|n-1|$ should not exceed 0.1 for any h or Ω_ν . Dvali et al. (1994) have analysed the Ω_ν - n plane for three values of h and found the same trends evident in Pogosyan & Starobinsky (1995a), although the limits on n were somewhat dependent on h as $0.80 \lesssim n(h/0.5)^{1/2} \lesssim 1.15$. An analysis solely of microwave anisotropies on various scales applied to tilted CHDM models (de Gasperis, Muciaccia & Vittorio 1995) favours low n values.

Our present paper is closest in spirit to the Pogosyan & Starobinsky (1995a) and Dvali et al. (1994) analyses, so it is worth indicating here where we differ. We make an entirely new calculation of the transfer functions for our models, including the incorporation of the baryonic component (not included by them) which is significant especially for low h values. In addition to using the more modern *COBE* normalization, we make a recalculation of constraints from cluster abundance, which are now more conservative. We include a treatment of damped Lyman alpha system abundance, which has been seen as problematic for some versions of the CHDM scenario. We shall also use results from the POTENT analysis of velocity fields (Bertschinger & Dekel 1989; Dekel 1994), but we include detailed modelling of the effects of cosmic variance (as included in Schaefer & Shafi 1994). Also, only $h > 0.4$ has been considered previously. Regardless of one's view regarding constraints from direct measurement, it is worth extending this to lower values to investigate the 'volume' of favoured parameters. Further motivation for this arises as models with extra massless species or decaying particles can mimic low values of h while keeping the actual h , as would be directly measured, higher (Dodelson, Gyuk & Turner 1994; White, Gelmini & Silk 1995b).

An important subset of the parameter space which we shall also explore is the case of pure CDM models: that is, the case $\Omega_\nu = 0$. These have seldom been studied in the context of permitting full variation of n , h and the gravitational wave amplitude, and it has recently been suggested (White et al. 1995b) that claims that all such models are ruled out may be premature. Our results support the assertion that there remains some viable parameter space for CDM models, without one having to modify the dark matter content or change the number of massless species.

It is useful to have a fiducial model to make comparisons with. We shall adopt the usual practice of taking this to be the standard CDM (SCDM) model, even though this is known not to be a good fit to the data. To be explicit about our assumptions, the parameters of this model are $n = 1$, no gravitational waves, $h = 0.5$, $\Omega_B = 0.016h^{-2} = 0.064$, $\Omega_\nu = 0$ and the amplitude of the spectrum normalized to match the four-year *COBE* observations with expected quadrupole $Q_{\text{rms-PS}} = 18.0 \mu\text{K}$ (Górski et al. 1996).

The layout of the paper is as follows. In Section 2 we briefly outline the derivation of the power spectra we use to make the comparison with the observations, along with some discussion of our use of Press-Schechter theory. In Section 3 we shall provide a detailed account of the observations we have selected in order to make comparison with the theoretical predictions. Our procedure is not to use the power spectrum itself, but instead to concentrate on the spectrum filtered by a top-hat window, which represents the variance of fluctuations on a given scale. This quantity has several advantages, and in Section 3 we shall describe how we interpret our chosen observations in terms of this quantity. Section 4 will then provide the confrontation of the theoretical predictions with the observations.

2 THE THEORETICAL INPUT

2.1 Transfer functions

Inflation generates Gaussian density perturbations, which implies that their stochastic properties can be completely described by the power spectrum. In almost all inflationary models, the power spectrum $P(k)$ can be accurately parametrized across observable scales by a power-law $P(k) \propto k^n$, where k is the comoving wavenumber (see Liddle & Lyth 1993a and references therein). The choice $n = 1$ gives the scale-invariant Harrison–Zel’dovich spectrum, but different inflationary models predict different n , with the overall range easily encompassing all values of n of interest for structure formation. Inflation will also generate long-wavelength gravitational waves which may contribute to the *COBE* signal; these will be discussed later.

We shall use a slightly different definition of the spectrum from the usual one, defining the spectrum of any type of perturbation f as (Liddle & Lyth 1993a)

$$\mathcal{P}_f(k) = 4\pi (Lk/2\pi)^3 \langle |f_k|^2 \rangle, \quad (1)$$

where L is the comoving size of the periodic box introduced to allow a Fourier expansion of f into its comoving modes f_k , and the angled brackets indicate an averaging over a small region of k -space to make the spectrum a smooth function. We have used statistical isotropy to say that the spectrum can only be a function of the magnitude of k , and not of its direction. The prefactor is chosen to guarantee that the mean square perturbation is given by

$$\sigma_f^2 = \int_0^\infty \mathcal{P}_f(k) \frac{dk}{k}. \quad (2)$$

Primarily we are interested in the spectrum \mathcal{P}_δ of the density contrast δ , which is related to the usual $P(k)$ by $\mathcal{P}_\delta \propto k^3 P(k)$.

The initial spectrum generated by inflation will be modified as the universe evolves, since the growth of density perturbations is affected by the properties of the matter in the universe and also the value of the Hubble parameter. This modification is quantified by the *transfer function* $T(k, z)$, which measures the amount of growth that a perturbation on scale k receives by a redshift z relative to the infinite wavelength $k = 0$ mode (thus $T(k, z) \rightarrow 1$ on large scales). With a power-law initial spectrum from inflation, at a given redshift one has

$$\mathcal{P}_\delta(k, z) \propto k^{3+n} T^2(k, z), \quad (3)$$

where the constant of proportionality is to be fixed via observations.

For reasons discussed in the next section, we choose not to try to place constraints directly on the power spectrum. Instead, we choose the dispersion $\sigma(R)$ of the density contrast smoothed on a scale R as our fundamental quantity. The smoothing is carried out via a top-hat window function, defined by

$$W(kR) = 3 \left(\frac{\sin(kR)}{(kR)^3} - \frac{\cos(kR)}{(kR)^2} \right), \quad (4)$$

which filters out modes with $k^{-1} \ll R$. The variance of the smoothed field is

$$\sigma^2(R) = \int_0^\infty W^2(kR) \mathcal{P}_\delta(k) \frac{dk}{k}. \quad (5)$$

Often the spectrum is increasing towards short scales, in which case the variance is dominated by modes with $k^{-1} \sim R$.

It is often useful to associate a mass with the top-hat filter, which one gets by integrating the filter over a uniform density. Assuming critical density, this yields

$$M(R) = 1.16 \times 10^{12} h^{-1} \left(\frac{R}{h^{-1} \text{Mpc}} \right)^3 M_\odot. \quad (6)$$

We have calculated the transfer functions numerically using the techniques described by Schaefer & de Laix (1996). The procedure can be summarized as follows. Starting from adiabatic initial conditions deep within the radiation-dominated epoch, the gauge-invariant linear evolution equations for each of the components are numerically integrated up to the present time via a Hanning-type Predictor–Corrector. We keep 1000 moments of the photon and relativistic neutrino distribution up until well into the matter-dominated epoch redshift $z = 250$, at which time we set their amplitudes equal to zero. At this time they have a negligible influence on the growth of the matter perturbations. The massive neutrinos require special treatment. In this case we expand the neutrino distribution function in terms of angular moments of the cosine of the angle between the particle momentum and the wavevector, keeping 200 angular moments. The massive neutrino distribution function must be integrated over momentum at every integration step, and this is done with 20-point Gauss–Laguerre integration which is accurate to better than one part in 10^6 . The photons and baryons are treated using the tight coupling approximation until the temperature drops below 6000 K, at which point we switch to the full equations for the two coupled components.

We calculate transfer functions for $\Omega_\nu = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$ for $h = 0.3, 0.4, 0.5, 0.6, 0.7$ using a value of the baryon fraction consistent with nucleosynthesis^{*}, $\Omega_B h^2 = 0.0125$. We calculate them for $z = 0, 3, 3.5, 4$. We have fit them with coefficients in a form somewhat similar to the Bardeen et al. (1986) CDM transfer functions; however, the coefficients are not smooth functions of Ω_ν , which proved to be inconvenient for testing. We note that there already exists a ‘universal’ transfer function for the CHDM models in universes with no baryons added (Pogosyan & Starobinsky 1995a), which we shall adapt to models with baryons. Pogosyan & Starobinsky’s transfer function begins with the Bardeen et al. (1986) fit to standard CDM, which is

$$T_{\text{SCDM}}(q) = \frac{\ln(1 + 2.34q)}{2.34q} \times [1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{-1/4}, \quad (7)$$

where the scaled wavenumber q is related to the usual Fourier wavenumber k as $q = k/h^2$. Pogosyan & Starobinsky (1995a) then constructed a formula for the factor that describes the damping of the massive neutrino component:

^{*} We carried out these tests using the old nucleosynthesis value of Walker et al. (1991), before our decision to adopt the higher value $\Omega_B h^2 = 0.016$ (Copi et al. 1995a,b) which is that used to obtain all results in this paper. This change does not affect our tests of the fitting quality.

$$D(q, z) = \left[\frac{1 + (Aq)^2 + a_{\text{eq}}(1+z)(1-\Omega_\nu)^{1/\beta}(Bq)^4}{1 + (Bq)^2 - (Bq)^3 + (Bq)^4} \right]^\beta, \quad (8)$$

where

$$\begin{aligned} \beta &= \frac{5}{4}(1 - \sqrt{1 - 24\Omega_\nu/25}); \\ A &= 17.266 \frac{(1 + 10.912\Omega_\nu)\sqrt{\Omega_\nu(1 - 0.9465\Omega_\nu)}}{1 + (9.259\Omega_\nu)^2}; \\ B &= 2.6823 \frac{1.1435}{\Omega_\nu + 0.1435}; \\ a_{\text{eq}} &= \frac{4.212 \times 10^{-5}}{h^2}. \end{aligned} \quad (9)$$

A widely used empirical formula for adding the effect of baryons to the standard CDM transfer function is to generalize the formula for q to $q = k/h\Gamma$ where the ‘shape parameter’ Γ is defined by $\Gamma = h \exp(-2\Omega_B)$. This is known to work well for CDM provided that Ω_B is not too large (Peacock & Dodds 1994). We then tested whether or not this worked with the Pogosyan & Starobinsky transfer functions. We found that this replacement works extremely well provided that $h \gtrsim 0.5$ and $\Omega_B \lesssim 0.1$. As $h \rightarrow 0.21$ the decoupling time approaches the time of matter–radiation equality, so the damping of fluctuation growth by the baryons recedes in importance. We have found that, for $h > 0.21$, a better replacement is to use

$$\Gamma = h \exp[-2(1 - (0.21/h)^2)\Omega_B], \quad (10)$$

which is very accurate for $\Omega_B \lesssim 0.1$. This relation also holds for pure CDM transfer functions. If $\Omega_B \gtrsim 0.1$, the baryons then become dynamically significant and impose a steep drop at the decoupling length scale, a feature which cannot be adequately described by simply shifting the scales in the transfer function. For $h = 0.3$, we have the central value $\Omega_B = 0.139$ and notice significant departures of the scaled Pogosyan & Starobinsky transfer function from our computed functions. We can compare values of the dispersion $\sigma(R)$ calculated using the real and the scaled transfer functions. In this case we find that the scaled functions overestimate the amplitude $\sigma(R)$ by as much as 10 per cent on small scales $\sim 1h^{-1}$ Mpc and underestimate it on scales $\sim 200h^{-1}$ Mpc. For comparison, when $h = 0.4$, implying $\Omega_B = 0.078$, the error in $\sigma(R)$ is less than about 2 per cent when $R > 0.1h^{-1}$ Mpc. At larger values of the Hubble constant the fits are even better.

Using the value $\Omega_B h^2 = 0.016$ that we adopt to obtain results in this paper, the accuracy of the scaled Pogosyan & Starobinsky transfer function becomes worse for small values of h ; a good fit to our computed functions across the range of scales that we are interested in can only be achieved for $h > 0.4$, instead of the previous limit $h > 0.35$. However, for Hubble constant values as low as these we find that using the exact transfer functions leads to slightly stronger constraints, so adopting the fitting function as above is a conservative choice.

Putting all this information together, the redshift-dependent transfer function for CHDM models is given by

$$T(k, z) = T_{\text{SCDM}}(k)D(k, z) \quad ; \quad q = k/h\Gamma, \quad (11)$$

where Γ is given by equation (10).

Some of the observations that we use apply at moderate redshift rather than redshift zero. In a cold dark matter

dominated universe this can easily be accounted for using the scale-independent linear growth law $\sigma(R) \propto (1+z)^{-1}$, implying a redshift-independent transfer function at late times. By contrast, in CHDM models the growth rate becomes scale-dependent with suppression on short scales due to neutrino free-streaming. Fig. 1 illustrates the redshift dependence of the transfer function for two choices of HDM density. We see that, on scales greater than $3h^{-1}$ Mpc, the CDM growth law is an excellent approximation from moderate redshift even when a sizeable HDM component is present.

2.2 Gravitational waves from inflation

In addition to generating a power-law spectrum of density perturbations, inflation generates a power-law spectrum of gravitational wave modes (Starobinsky 1979; Liddle & Lyth 1993a). The only observation we discuss that these are capable of influencing is the *COBE* observation, where a possible gravitational wave contribution to microwave background anisotropies (Abbott & Wise 1984; Starobinsky 1985) will add in quadrature to that from density perturbations.

Within the usual slow-roll inflation models, the amplitude of gravitational waves on *COBE* scales is another free parameter, independent of the spectral index of density perturbations[†] (Liddle & Lyth 1992). We shall treat the amplitude as given independently; the independent choice of n and the gravitational wave amplitude is then the most general outcome of slow-roll inflation for any choice of potential for the scalar field driving inflation (Liddle & Lyth 1993b).

If one were to be more specific in the choice of inflation model, then the spectral index and gravitational wave amplitude could be related. For example, power-law inflation yields $n < 1$ and $r \simeq 2\pi(1-n)$, where r is the relative contribution of gravitational waves to density perturbations to large-angle microwave background anisotropies[‡], as defined by Liddle & Lyth (1992). Almost all known inflation models have gravitational wave contributions sandwiched between zero and that of a power-law inflation model yielding the same spectral index. We shall concentrate on these two options for $n < 1$, and ignore gravitational waves for $n > 1$ since it is hard to make inflationary models with $n > 1$ and significant gravitational waves.

2.3 Press–Schechter theory

The standard comparisons that we make between theory and observations, based on the spectrum integrated with a top-hat filter, are well established in the literature. The exception is the calculations based on object abundance, which contain greater theoretical uncertainties than other

[†] However, the spectral index of the gravitational wave spectrum is then related to the amplitude via a ‘consistency relation’.

[‡] In some papers, the relative amplitude of gravitational waves and density perturbations is given as $7(1-n)$. This refers to the relative contributions to the quadrupole, which has a correction from the curvature of the last scattering surface. The version we give is appropriate to higher multipoles, and since the *COBE* normalization is most sensitive around the tenth multipole it is the best version to use in this context.

measures, and so we shall discuss in depth the way that we carry this out. The standard technique is Press–Schechter theory (Press & Schechter 1974), which has been compared in depth with N -body simulations (e.g. Lacey & Cole 1993, 1994), and we shall use it to obtain constraints on the abundances of each of damped Lyman alpha systems, quasars and galaxy clusters.

When one applies a smoothing window with a given radius to a Gaussian random density field, one obtains the corresponding *smoothed* density field which is also a Gaussian random field provided that its dispersion is smaller than one. It is then straightforward to obtain the fraction of space in the universe occupied by regions where the *linearly* evolved smoothed density contrast exceeds some given threshold value. The insight of Press & Schechter was to assume that for the correct threshold value this fraction could be identified with the fraction of matter in the universe that is part of gravitationally bound objects with a certain minimum mass, the relation between the size of the regions and the minimum mass of the bound objects depending on the smoothing window applied to the underlying density field.

A problem with this assumption is that in linear theory half the volume of the universe is always composed of regions with a negative smoothed density contrast, and therefore only half of all the matter in the universe is available to form bound structures, which clearly does not happen in the real Universe. This problem arises because one is not taking into account the matter in the regions whose linearly evolved density contrast does not exceed the threshold value, and thus are not considered to be bound according to the above criterion, but which are part of bigger regions whose linearly evolved density contrast does exceed the threshold value, and are therefore bound. The original Press–Schechter derivation tries to allow for the matter in those regions simply by assuming that they contain as much matter as is contained within the regions that are bound according to the original criterion. Though this assumption makes some sense if one thinks in terms of the statistics of a Gaussian random field, the main motivation was that it is the simplest way of allowing all the matter in the universe to be available to form gravitationally bound structures. This is less than satisfactory, and since then many people have tried in all sort of ways to determine if this assumption has any validity. The conclusion reached from N -body simulations is that it depends on the smoothing window, being a reasonable assumption for a window that is a top-hat in k -space, known as a sharp- k window (for which Peacock & Heavens (1990) and Bond et al. (1991) have proven using analytic methods that the factor two correction is exact[§]) and for the real space top-hat window that we use, but not so good for a Gaussian window (Lacey & Cole 1994). We use the top-hat window as the relation between the size of a region and its mass is then straightforward, which is not the case for the sharp- k window.

The density in collapsed objects above a given mass at a redshift z is then given simply by integrating over the tail

of the Gaussian with the additional factor two multiplier, yielding

$$\Omega(> M(R), z) = \text{erfc} \left(\frac{\delta_c}{\sqrt{2} \sigma(R, z)} \right), \quad (12)$$

where δ_c is the threshold value, $\sigma(R, z)$ is the dispersion smoothed on scale R at redshift z and ‘erfc’ is the complementary error function.

The choice of threshold is crucially important, as typically it is one of the main sources of uncertainty. The literature features a wide range of values, but it is vital to note that this is primarily because different types of smoothing window require different thresholds. Once a specific choice of window is made the uncertainty is not so great. In the original manifestation of the Press–Schechter theory, a threshold δ_c of 1.7 was motivated by the spherical collapse model for a top-hat perturbation. However, this is a highly idealized model which assumes that the collapsing perturbation is not under any external influence: that is, it does not possess shear. This should be an increasingly good assumption the less evolved the smoothed density field is (Bernardeau 1994) and the less relative large-scale power there is. The influence and relative importance of shear on the time a perturbation takes to collapse depends critically on one’s definition of collapse. If one identifies collapse of a perturbation with collapse along the first collapsing axis then shear decreases the time-scale of the collapse, but if one identifies collapse of a perturbation with complete collapse along three normal axes then shear increases this timescale (Monaco 1995). The first case relates to the formation of pancakes, and for example can be useful in the study of the objects that give rise to the Lyman alpha forest lines in the spectra of quasars. However, if one is interested in completely virialized objects like quasars or clusters then the second definition of collapse should be used. The damped Lyman alpha systems are likely to lie somewhere in between these two extremes.

To a large extent, the analytic modeling of the threshold has been superseded by direct calibration of the Press–Schechter theory with N -body simulations. Indeed, if one were to take an extreme view one could regard the Press–Schechter formula simply as a fitting function to the number density at a given epoch. Comparison with N -body simulations indicates that for the top-hat filter the spherical collapse estimate $\delta_c = 1.7$ actually works extremely well for virialized objects, with at most an uncertainty of 0.2 in either direction (Lacey & Cole 1994).

It is often emphasized that the predicted number density can depend very sensitively on the choice of threshold and on the dispersion, especially where the dispersion is small. This is of great advantage for this application, because it means that, even if there is a large observational uncertainty in the number density, this gives only a small uncertainty in the estimate of the dispersion. Concerning the uncertainty in the threshold, we can see directly from the Press–Schechter formula above that an uncertainty of 12 per cent in δ_c translates into the same uncertainty in the estimate of $\sigma(R)$.

[§] Recently Yano, Nagashima & Gouda (1995) have recovered this result using a different technique, first proposed by Jedamzik (1995), which relies on the use of the integral equation of the mass function.

3 THE OBSERVATIONAL DATA

To constrain the density perturbation spectrum effectively, one requires a compilation of estimates of its amplitude at a variety of different scales by a variety of different methods. To some extent, this occurs naturally as different types of observations are best suited to estimating the power spectrum on different scales. For example, only microwave background data are presently capable of providing information on the largest scales, and only the abundance of objects at high redshift allows access to presently non-linear scales at a time when they may still be addressed using quasi-linear theory. It is only the intermediate scales, running from perhaps $8h^{-1}$ Mpc up to $100h^{-1}$ Mpc, that have been simultaneously constrained by a number of different types of measurements, from abundance of clusters to galaxy correlation functions to peculiar velocity flows; in the near future reliable microwave background experiments should also extend down into this region.

Our general strategy is not to impose constraints on the power spectra $P(k)$ directly (where k is the comoving wavenumber), but instead to impose them on the dispersion of the density field filtered through a top-hat window function, denoted $\sigma(R)$, whose radius R is varied in order to pick out different scales. This method is useful because the bulk of the observational data are obtained in this form, and typically the conversion of such data into power spectrum form introduces systematic errors. In contrast, a theoretical calculation of the filtered variance is very simple to make given a theoretical power spectrum. Further, observations on short scales connected with object formation at high redshift have no interpretation at all in terms of the power spectrum at a given wavenumber; the standard method of theoretical comparison using the Press–Schechter calculation deals directly with the filtered variance. Concentrating on estimating a single function such as this has the advantage that to a large extent the data can be presented together and treated on the same footing.

3.1 COBE

Recently there has been considerable activity concerning the interpretation of the anisotropies detected by *COBE* (Smoot et al. 1992; Bennett et al. 1994; Wright et al. 1994). The four-year data set is now available (Bennett et al. 1996; Banday et al. 1996; Górski et al. 1996; Hinshaw et al. 1996) and we will use results from it in this paper. The methods used have become sufficiently sophisticated that simple normalization methods, such as to the 10° variance of the anisotropies as widely used in the two years following the *COBE* announcement, are no longer appropriate, since they make inadequate use of the full *COBE* data set. Instead, it is better to rely on the normalizations published in the literature which do take the full data set into account.

The first development in this regard was a very elegant pair of papers by Górski and collaborators (Górski 1994; Górski et al. 1994) who fitted power-law spectra to the observations, thus obtaining likelihoods in the n – $Q_{\text{rms-PS}}$ plane, where $Q_{\text{rms-PS}}$ is the expected quadrupole (over an ensemble of independent observers). The quantity that is actually desired in order to constrain theoretical models is not the full likelihood or the marginalized one, but rather the condi-

tional likelihood on $Q_{\text{rms-PS}}$ for fixed n , given as a function of n . Although they only provided this for $n = 1$, they noted that regardless of the fitted n the preferred amplitude of the ninth multipole remains unchanged to excellent accuracy, and this result can be used to generate the required normalization as a function of n .

However, more recently it has been noted that the assumption of a power-law spectrum of anisotropies, corresponding to the Sachs–Wolfe contribution, is not a perfect one, because the ‘Doppler peak’ extends to small multipoles and invades part of the region that *COBE* samples. Consequently, one should fit the amplitude using full anisotropy spectra. This was carried out for the two-year *COBE* data by Bunn, Scott & White (1995). For CDM spectra with no gravitational waves, they find conditional likelihoods yielding

$$Q_{\text{rms-PS}}(n) = (19.9 \pm 1.5) \exp[0.69(1 - n)] \mu\text{K} \quad (2\text{yr}). \quad (13)$$

Note that the n dependence in the fit given was calculated only taking into account the Sachs–Wolfe effect; it should nevertheless provide a very good approximation. Bunn et al. (1995) noted that this result is more or less independent of the nature of any dark matter, of Ω_B and of h , so it can be used for all models without gravitational waves. Although the full anisotropy spectra are needed for performing the fit to the *COBE* data, it is fine to compute the perturbation spectrum normalization corresponding to a given quadrupole using the Sachs–Wolfe formula, since the quadrupole is least affected by the Doppler peak.

We need to correct this for the new four-year data, for which amplitudes conditional on n have not yet been published. However, the principal change, due largely to a new galactic cut strategy, is a lowering of the normalization without changing the shape information. It is therefore fine to use the same n -dependence with the lowered normalization (Górski et al. 1996; M. White, private communication), yielding

$$Q_{\text{rms-PS}}(n) = (18.0 \pm 1.4) \exp[0.69(1 - n)] \mu\text{K} \quad (4\text{yr}). \quad (14)$$

This is the normalization that we shall adopt. The quoted error is 1σ .

As we are concentrating on interpreting data in terms of the filtered dispersion $\sigma(R)$, it is interesting to ask what sort of scales this normalization is sampling. One way to do this is to normalize a set of CDM models with different n and see where the curves cross. One finds that the lines more or less cross (with an accuracy of a few per cent, doing less well as h is varied) at a scale of $4000h^{-1}$ Mpc. We shall occasionally use this to represent the *COBE* data schematically, with the main purpose of indicating the size of the *COBE* error; however, in all cases we shall calculate using the precise normalization of the power spectrum given above rather than this approximate data point.

To be completely accurate, if gravitational waves are included one should add their radiation power spectrum to that of the density perturbations and perform a full model fit. However, as long as the gravitational wave contribution is not too significant, one can approximate it as having the same functional form as the density perturbations over the *COBE* range and simply normalize down the density perturbation power spectrum as appropriate. As discussed by Liddle & Lyth (1993b), the relative contribution of grav-

itational waves to density perturbations to the microwave anisotropies, r as defined in Subsection 2.2, leads to a reduction in the amplitude of the *COBE* normalized dispersion $\sigma(R)$ by a factor $1/\sqrt{1+r}$.

Although we will normalize $\sigma(R)$ to the Górski et al. (1996) *COBE* central value, we shall allow for their 15 per cent uncertainty at the 2σ level by adding it in quadrature to the relative errors of those other observations which also constrain the amplitude of $\sigma(R)$.

3.2 Galaxy correlations

The deficiencies in the shape of the standard CDM spectrum are most apparent in surveys of galaxy correlations spanning the range from a few megaparsecs up to tens of megaparsecs. A variety of surveys such as QDOT, CfA, APM and 1.2 Jansky provide information in this region. In an attempt to evade systematics particular to the types of analysis provided, one can combine data from a variety of different sources, hoping to demonstrate consistency between the different data sets, and this has been achieved in an impressive analysis by Peacock & Dodds (1994). The cost is that the formal errors are somewhat larger than those one sees in individual surveys, and it is not easy to see whether or not one is unfairly penalizing the most accurate data sets rather than uncovering overoptimistically small error bars across all data sets.

Another problem with using galaxy data is that, although they determine the shape of the spectrum very well, the overall normalization is less certain due to the expectation that galaxy correlations are biased relative to the underlying matter, multiplying the power spectrum by a (hopefully scale-independent at least over the limited range of scales considered) bias parameter. One can attempt to determine the bias parameter from the surveys themselves by using redshift distortions and/or non-linear effects, or instead by utilizing an entirely separate method such as peculiar velocity flows. Alternatively one can allow the normalization of the galaxy correlation data to ‘float’, with its best amplitude determined by the other types of data under consideration, which amounts to throwing away information on the bias.

Peacock & Dodds (1994) quote their final results in terms of the power spectrum $\mathcal{P}_\delta(k)$ ($\Delta^2(k)$ in their notation). However, the original data are provided in a mixture of the power spectrum, the dispersion $\sigma(R)$ and the correlation function $\xi(R)$. They switch between them using an analytic prescription:

$$\sigma(R) = \mathcal{P}_\delta^{1/2}(k_R); \quad (15)$$

$$\xi(R) = \mathcal{P}_\delta^{1/2}(\sqrt{2}k_R), \quad (16)$$

where

$$k_R = \left[\frac{1}{2} \Gamma \left(\frac{m+3}{2} \right) \right]^{1/(m+3)} \frac{\sqrt{5}}{R}, \quad (17)$$

and $m \equiv (k/\mathcal{P}_\delta)(d\mathcal{P}_\delta/dk)$ is the effective spectral index. These formulae are obtained by assuming m constant over the range of k modes contributing, and using the approximation

$$W(kR) = \exp(-k^2 R^2/10), \quad (18)$$

which is exact for $kR \ll 1$.

In making the conversion, one needs to specify a power spectrum in order to calculate the effective spectral index. This is best done by choosing a model spectrum that fits the data; we use the best-fitting CDM spectrum, specified by a shape parameter Γ . Since the raw data are provided in a variety of forms, there is no reason to think that expressing them in terms of $\sigma(R)$ is any less accurate than expressing them via the power spectrum, and we shall use both. The shape parameter provides a good indication of the quality of fit to the galaxy correlation data and we shall occasionally use that language.

Recently some doubt has been cast over the assumption by Peacock & Dodds that the bias parameter is scale-independent down to the smallest scales, around $4h^{-1}$ Mpc, considered in their analysis (Peacock 1996). As it seems that it is for scales below around $8h^{-1}$ Mpc that the bias parameter starts becoming non-linear, we have excluded from their final data, presented in table 1 of Peacock & Dodds (1994), the four points corresponding to the smallest scales and re-calculated the best Γ fit to their remaining data. For $0.7 < n < 1.2$ we find $\Gamma = 0.23 - 0.28(1 - 1/n)$, where the 2σ relative error is +18 per cent and -15 per cent. This compares with the central value (for $n = 1$) from the full data set of 0.255 (Peacock & Dodds 1994).

When utilizing data of this form in a statistical analysis, as we do below, it is vital to ensure that the points used are taken suitably far apart as to be independent, and in general one needs the full correlation matrix to determine this (which has been calculated only for the QDOT survey power spectrum (Feldman, Kaiser & Peacock 1994)). If one is not careful as regards this point, then statistical tests are biased and, depending on the form of test used, this can make bad models look good or, much more seriously, make good models look bad. For a statistical treatment there is the further problem that the errors are systematic as well as statistical, and hence will not be normally distributed; unfortunately in the absence of a detailed understanding of an experiment there is no way to counter this other than to treat results with mild scepticism.

After the exclusion of the four points corresponding to the smallest scales, a chi-squared analysis of the remaining data in table 1 of Peacock & Dodds (1994), where n , h , Ω_ν and the normalization are the fitting parameters, has 8 degrees of freedom. Performing this analysis we find a very low minimum chi-squared of around 4; although it is perfectly reasonable that this occurred by chance, it may also indicate weak residual correlations of neighbouring data points. In the present case this typically makes models seem much better in relation to the data than they really are. The best way that we found of avoiding this problem is to calculate not the absolute exclusion level of each model against the data, but the relative confidence limits in the three-dimensional space formed by the parameters n , h and Ω_ν (Press et al. 1992). This is achieved by calculating the difference between the chi-square obtained for each model characterized by a fixed set of values for n , h and Ω_ν , where the normalization is calculated so as to minimize the chi-squared, and the minimum chi-squared obtained by varying the four parameters. This difference still has a chi-squared distribution, now with three degrees of freedom. The 68 per cent and 95 per cent confidence limits are then defined in the (n, h, Ω_ν) space by

the chi-squared difference being respectively smaller than 3.508 and 7.815. We will plot cross-sections of the region in the (n, h, Ω_ν) space defined by the 95 per cent confidence limit.

3.3 Peculiar velocities

3.3.1 POTENT

Peculiar velocities directly sample the matter power spectrum and so are unaffected by clustering bias. However, measurements of the peculiar velocity field are much harder to obtain. The best measurements using velocities alone come from the POTENT method (Bertschinger & Dekel 1989), the most recent version available being the Mark III POTENT data (Dekel 1994), which supply an estimate of the velocity smoothed on various length scales around us. This can be used as an estimator for $\sigma(R)$ on a particular scale, as follows.

First, we restrict ourselves to using a single piece of data, the velocity on a $40h^{-1}$ Mpc sphere. Although measurements exist for a range of scales, they are very highly correlated because the window function for the peculiar velocities samples a wide range of scales and in particular is more sensitive to longer scales than the density dispersion. We choose this particular value as it is in the centre of the supplied range.

In making a theoretical comparison, one needs a two-stage smoothing, since POTENT involves first smoothing the observed peculiar velocities with a $12h^{-1}$ Gaussian before the velocity reconstruction can be undertaken and the $40h^{-1}$ top-hat smoothing applied to obtain $v(40h^{-1}\text{Mpc})$. The appropriate formula for the dispersion of the velocity is

$$\sigma_v^2(R) = H_0^2 \int_0^\infty W^2(kR) \exp(-(12h^{-1}k)^2) \frac{\mathcal{P}_\delta}{k^2} \frac{dk}{k}. \quad (19)$$

As with *COBE* above, one can then ask what scales in the filtered dispersion $\sigma(R)$ of the density field correspond to a fixed observed velocity. This can again be addressed by plotting $\sigma(R)$ for a set of CDM models with different n , each normalized to yield the same $\sigma_v(40h^{-1}\text{Mpc})$. It turns out that such curves cross, extremely accurately, at a scale of $113h^{-1}$ Mpc. As stated above, the velocities sample considerably longer scales than the smoothing length by itself suggests.

This crossing point remains quite accurate even if one goes to CHDM models, and this fact coupled with the much larger observational errors as compared with *COBE* means that we can represent the POTENT data as a single constraint on $\sigma(113h^{-1}\text{Mpc})$.

The Mark III POTENT analysis gives for the bulk flow in a $40h^{-1}$ Mpc sphere (Dekel 1994)

$$v_{\text{POTENT}}(40h^{-1}\text{Mpc}) = 373 \pm 50 \text{ km s}^{-1}, \quad (20)$$

where the error arises from different ways of dealing with sampling-gradient bias and can thus be thought of as reflecting the systematic uncertainty in the POTENT analysis. Additionally there is an intrinsic uncertainty in the POTENT calculation due to random distance errors, which at the 1σ level is $\simeq 15$ per cent (Dekel 1994). Note that *COBE* normalized standard CDM yields

$$v_{\text{SCDM}}(40h^{-1}\text{Mpc}) = 409 \text{ km s}^{-1}, \quad (21)$$

suggesting that SCDM produces about the right answer using the modern *COBE* normalization. The observational error is dominated by cosmic variance, resulting from the POTENT observation being a single measurement from a random field. Since each velocity component separately has a Gaussian distribution, the velocity squared has a chi-squared distribution with three degrees of freedom. From this one can calculate the range of theoretical values for which the observed value would not lie in the tail of the distribution, and the probabilities corresponding to 68 per cent confidence yield an upward error of 89 per cent and a downward error of 24 per cent on the estimator for $\sigma(R)$. At the 95 per cent confidence level the error bars are +273 per cent and -43 per cent. The asymmetry of the errors originates in the asymmetry of the chi-squared distribution. We can now convolve the systematic and random errors arising from the POTENT calculation with the cosmic variance error. Assuming that the error in expression (20) corresponds to something like 95 per cent confidence (though as it is the smallest error this assumption is insignificant), we then obtain the total error in using the Mark III POTENT bulk flow calculation as an estimator of the normalization of the dispersion of the density contrast: at the 68 per cent confidence level, +98 per cent and -25 per cent; at the 95 per cent confidence level, +295 per cent and -47 per cent. Clearly, only the lower limits are of use for us. At a level corresponding to 95 per cent confidence, the bulk flow constraint can then be written as

$$\frac{\sigma_{\text{POTENT}}(113h^{-1}\text{Mpc})}{\sigma_{\text{SCDM}}(113h^{-1}\text{Mpc})} = 0.91_{-47 \text{ per cent}}^{+295 \text{ per cent}}. \quad (22)$$

3.3.2 Velocities versus densities

An alternative use of velocity data is through the comparison with the density field obtained via galaxy surveys. Present technology focuses on an estimate of a single parameter $\Omega_0^{0.6}/b$, where b is the bias parameter appropriate to whatever type of galaxies is being studied, normally *IRAS* galaxies with bias b_I . The degenerate combination of Ω_0 and b arises through the inability to distinguish slow velocities due to a slowing of the perturbation growth rate in low-density universes from having a high irregularity in the galaxy distribution relative to that of the matter distribution actually generating the velocities. However, we are considering only critical-density models, so these methods directly estimate the bias. This information can then be used in conjunction with the galaxy number counts dispersion to supply constraints on the variance in the density. Note though that there seems no good way to quantify the errors arising from the inadequacy of a single bias parameter to explain the difference between the galaxy and density variances.

We shall not utilize the range of bias found by Peacock & Dodds (1994), the reason being that there remains widespread disagreement in the literature between values obtained by different methods (for instance, see Dekel 1994 for a compilation). Consequently, the true uncertainty appears much greater than advertized by any single study, and if one attempts to take a more realistic view the amplitude becomes so uncertain as to provide no useful constraint.

3.4 Abundance of galaxy clusters

The typical mass of large galaxy clusters, about $10^{15} M_{\odot}$, corresponds to a linear scale of around $8h^{-1}$ Mpc. Observation indicates that large clusters are relatively rare, suggesting that this scale is still in the quasi-linear regime. The usual technique of Press–Schechter theory calibrated by N -body simulations can therefore be used to impose constraints. A variety of estimates of the cluster mass function exist in the literature; some authors (e.g. Lilje 1992; White, Efstathiou & Frenk 1993a) aim to reproduce the observed number density at a single mass scale whilst others (e.g. Evrard 1989; Henry & Arnaud 1991; Hattori & Matsuzawa 1995) more ambitiously aim to fit the shape of the cluster mass function. The analysis we perform belongs to the first type. Typically, the number density of a given type of cluster is quite well known, at least at low redshift — most of the uncertainty comes from poor knowledge of the mass of individual clusters. This can be estimated in a variety of ways, the most common being the virial theorem, the X-ray temperature distribution as a tracer of the gravitational potential and, most recently, weak shear lensing of background objects. All these methods suffer from several problems, though the one that at the present seems most likely to give the best results is the use of X-ray temperature observations.

The observed number density of clusters per unit temperature at $z = 0$ about a mean X-ray temperature of 7 keV was calculated by Henry & Arnaud (1991) to be

$$n(7 \text{ keV}, 0) = 2.0^{+2.0}_{-1.0} \times 10^{-7} h^3 \text{ Mpc}^{-3} \text{ keV}^{-1}. \quad (23)$$

The comoving number density of clusters with virial mass M_v per mass interval dM_v at a redshift z is obtained by differentiating equation (12) with respect to the mass and multiplying it by ρ_b/M_v , where ρ_b is the comoving background density (a constant during matter domination), thus giving

$$n(M_v, z) dM_v = -\sqrt{\frac{2}{\pi}} \frac{\rho_b}{M_v} \frac{\delta_c}{\Delta^2(z)} \frac{d\Delta(z)}{dM_v} \exp\left[-\frac{\delta_c^2}{2\Delta^2(z)}\right] dM_v, \quad (24)$$

where $\Delta \equiv \sigma(r_L)$ with r_L the comoving linear scale associated with M_v , $r_L^3 = 3M_v/4\pi\rho_b$. Traditionally the cluster abundance is used to constrain the present-day dispersion at $8h^{-1}$ Mpc, $\sigma_8 \equiv \sigma(8h^{-1} \text{ Mpc}, 0)$, and the quantity Δ is specified by an analytic approximation to the power spectrum in the vicinity of this scale. Generally, one can write

$$\Delta(z) = \sigma_8(z) \left(\frac{r_L}{8h^{-1} \text{ Mpc}} \right)^{-\gamma(r_L)}. \quad (25)$$

In Liddle et al. (1996) we adopted the form

$$\gamma(r_L) = (0.3\Gamma + 0.2) \left[2.92 + \log\left(\frac{r_L}{8h^{-1} \text{ Mpc}}\right) \right], \quad (26)$$

where Γ is a shape parameter. Though this fit is strictly only correct for scale-invariant pure CDM models, it can also be used as a fitting function for the dispersion of the observed linear power spectrum on some restricted range of scales, which for our purposes means within a factor of 1.5 of $8h^{-1}$

Mpc. The values used for Γ will then be those allowed by observations, i.e. $\Gamma \in [0.19, 0.27]$ at the 2σ confidence level[¶].

Note that, unlike with pure CDM models, the shape of the power spectrum for CHDM models is not redshift independent since the growth of perturbations at a given scale depends on the mean random peculiar velocities of the massive neutrinos at the scale in question which in turn are redshift dependent. However, for the scales of interest for clusters, in the CHDM models that we consider the redshift evolution of the shape of the power spectrum is extremely small in the redshift interval where most clusters form in these models, i.e. $z \leq 0.5$.

Using expression (25) to calculate the derivative in equation (24), we therefore get

$$n(M_v, z) dM_v = \sqrt{\frac{2}{\pi}} \frac{\rho_b}{M_v^2} \frac{2.92(0.3\Gamma + 0.2)\delta_c}{3\Delta(z)} \exp\left[-\frac{\delta_c^2}{2\Delta^2(z)}\right] dM_v. \quad (27)$$

As we are considering clusters massive enough that at the corresponding scale the density field is not yet well developed into the non-linear regime, according to the discussion on Subsection 2.3 we can therefore ignore the influence of shear on their formation and assume that they collapsed spherically. Nevertheless, to be conservative we shall include an assumed 1σ dispersion of ± 0.1 in the value of δ_c , i.e. $\delta_c = 1.7 \pm 0.1$.

Using self-similar evolution arguments (e.g. Hanami 1993), which have been shown to be in good agreement with hydrodynamical N -body simulations (Navarro, Frenk & White 1995), one obtains the following relation between the cluster virial mass, M_v , its mean X-ray temperature, $k_B T$, and its redshift of virialization, z_c :

$$M_v \propto (1 + z_c)^{-3/2} (k_B T)^{3/2}. \quad (28)$$

We begin by considering the case of a CDM universe. In order to normalize equation (28) we use results from the hydrodynamical N -body simulations for an $\Omega_0 = 1.0$ CDM model performed by White et al. (1993b). From a catalogue of 12 simulated clusters with a wide range of X-ray temperatures they estimated that a cluster with a present mean X-ray temperature of 7.5 keV corresponds to a mass within one Abell radius ($1.5 h^{-1}$ Mpc) of the cluster centre of $M_A = (1.10 \pm 0.22) \times 10^{15} h^{-1} M_{\odot}$. The error arises from the dispersion in the catalogue and is supposed to represent the 1σ significance level. White et al. (1993b) also found that the simulated clusters had a density profile in their outer regions approximately described by $\rho_c(r) \propto r^{-2.4 \pm 0.1}$. This same result was obtained by Metzler & Evrard (1994) and Navarro et al. (1995). Bearing in mind that the cluster virial radius in a $\Omega_0 = 1.0$ universe encloses a density 178 times the background density, it is then straightforward to calculate the cluster virial mass from M_A . Through a Monte Carlo procedure, where we assume the errors in M_A and in the exponent of $\rho_c(r)$ to be normally distributed, we find $M_v = (1.23 \pm 0.32) \times 10^{15} h^{-1} M_{\odot}$ for a cluster with a present mean X-ray temperature of 7.5 keV in an $\Omega_0 = 1.0$ universe. Assuming that such a cluster virialized at a red-

[¶] Using the Peacock & Dodds (1994) 2σ interval, $\Gamma \in [0.22, 0.29]$, does not change the final results.

shift of $z_c \simeq 0.05 \pm 0.05$ (e.g. Metzler & Evrard 1994; Navarro et al. 1995), we can now normalize equation (28):

$$M_v = (1.32 \pm 0.34) \times 10^{15} \times (1 + z_c)^{-3/2} \left(\frac{k_B T}{7.5 \text{ keV}} \right)^{3/2} h^{-1} M_\odot. \quad (29)$$

This result is in very close agreement with the one obtained by Evrard (1990) from his own hydrodynamical N -body simulations. Hence the virial mass M_v for a cluster with a present mean X-ray temperature of 7 keV is given by

$$M_v = (1.2 \pm 0.3) \times 10^{15} (1 + z_c)^{-3/2} h^{-1} M_\odot. \quad (30)$$

Through some simple physical arguments, Sasaki (1994) used Press–Schechter theory to obtain an expression for the comoving number density of clusters per mass interval dM_v about virial mass M_v , which virialize in an interval dz about some redshift z and survive until the present:

$$N(M_v, z) dM_v dz = \left[-\frac{\delta_c^2}{\Delta^2(z)} \frac{n(M_v, z)}{\sigma_8(z)} \frac{d\sigma_8(z)}{dz} \right] \frac{\sigma_8(z)}{\sigma_8} dM_v dz, \quad (31)$$

where for the type of models we are presently considering we have

$$\sigma_8(z) = \sigma_8(1 + z)^{-1}. \quad (32)$$

In equation (31) the expression within the square brackets gives the formation rate of clusters with virial mass M_v at redshift z , whereas the fraction outside gives the probability of these clusters surviving until the present. If one now assumes that at each redshift z the cluster virial mass M_v in equation (31) is determined by expression (30) with $z_c = z$, then equation (31) gives the comoving number density of clusters *per unit mass* that virialize at each redshift z and survive up to the present such that they have a mean X-ray temperature of 7 keV at the present. Through the chain rule we can then determine the comoving number density of clusters *per unit temperature* that virialize at each redshift z and survive up to the present such that they have a mean X-ray temperature of 7 keV at the present:

$$\begin{aligned} N(k_B T, z) d(k_B T) dz &= \frac{dM_v}{d(k_B T)} N(M_v, z) d(k_B T) dz \\ &= \frac{3}{2} \frac{M_v}{k_B T} N(M_v, z) d(k_B T) dz, \end{aligned} \quad (33)$$

where the second equality uses equation (28). We therefore have

$$N(k_B T, z) d(k_B T) dz = \frac{3}{2} \frac{M_v}{k_B T} \frac{\delta_c^2}{\Delta^2(z)} \frac{n(M_v, z)}{(1 + z)^2} d(k_B T) dz. \quad (34)$$

Numerically integrating this expression from $z = 0$ to $z = \infty$ then gives the present comoving number density of clusters per unit temperature with a mean X-ray temperature of 7 keV as a function of the present value of σ_8 . Comparing with the observational value given by equation (23) we then find to a good approximation that

$$\sigma_8 = 0.60^{+0.19}_{-0.15}. \quad (35)$$

The errors in equation (35) represent 95 per cent confidence levels and arise from the dispersions in the observational value of Γ , in the assumed value for δ_c , and in expressions

(23) and (30). They were estimated via a Monte Carlo procedure, the full details of which are given by Viana & Liddle (1996). That paper also demonstrates that essentially the same constraint can be obtained using the more detailed merging picture due to Lacey & Cole (1993, 1994).

However, this result applies only to models where all the dark matter is cold. We would now like to know how this result is affected if one substitutes part of the cold dark matter by massive neutrinos.

In galaxy clusters the X-ray emission comes mainly from a nearly isothermal core, and thus strongly depends on the depth and width of its gravitational potential. Outside the core the shape of the gravitational potential is of much less importance to the total X-ray emission. It is then possible to have galaxy clusters with the same mean X-ray temperature at virialization but slightly different virial masses. Though for a given cosmological model this dispersion should be quite small, the differences in virial mass between galaxy clusters with the same mean X-ray temperature at virialization in two different cosmological models could be significantly higher.

Whilst the dependence of cluster density profiles on the slope of the power spectrum at the cluster scale has been studied quite thoroughly (Crone, Evrard & Richstone 1994), the consequences of changing the nature of some of the dark matter have not been so extensively studied. In the case of interest to us, where only one neutrino species has a cosmologically significant mass, the typical cluster density profile has been determined only for a pure neutrino model (Cen 1994) and for a model with $\Omega_\nu = 0.3$ (Kofman et al. 1995). As the fraction of massive neutrinos is increased at the expense of the same amount of cold dark matter, the depth and width of the gravitational potential at the nearly isothermal core, and therefore the core mass and radius, should remain approximately the same for clusters with equal mean X-ray temperature at virialization. However, we now have a component that clusters less, therefore leading to a more extended mass distribution. For a power-law density profile $\rho \propto r^{-\alpha}$, this corresponds to a smaller α . It can then easily be shown that the cluster virial mass increases. This increase will be greater either if more CDM is substituted by HDM or if the neutrino free-streaming length is increased by making them lighter^{||}. In reality these two effects oppose each other as the neutrino mass increases with Ω_ν . In the limit where all the dark matter is composed of massive neutrinos, these are sufficiently massive that, at the scales corresponding to high-mass clusters, the clustering behaviour of the massive neutrinos seems to resemble closely that of cold dark matter (Cen 1994). If Ω_ν is between 0 and 1, then for high-mass galaxy clusters we have very little information about the clustering properties of massive neutrinos on the

^{||} If the neutrino mass is so small that the neutrinos are unable to cluster at the scales we are considering, around $2h^{-1}$ Mpc, the cluster virial mass will not increase as the neutrinos will not be gravitationally bound to the cluster. However, the effect on the relationship between σ_8 and the cluster number density will be exactly the same as if the cluster virial mass had increased in reality, as will become clear in the section dealing with damped Lyman alpha systems.

scales in which we are interested, and therefore about the virial masses one should expect in such models.

To our knowledge there is only one hydrodynamical N -body simulation study (Bryan et al. 1994) that has tried to relate σ_8 to the abundance of X-ray clusters for a CHDM model. Though their resolution is insufficient to determine the internal density distribution of the galaxy clusters that they obtain, we can use the Press–Schechter approximation to re-normalize their simulation. First we need to calculate the present-day cluster virial mass M_v which corresponds to a present mean X-ray temperature of 7 keV by using their normalization of the power spectrum, $\sigma_8 = 0.606$, and the cluster number densities they obtain for that X-ray temperature. Using expression (34) and assuming $\delta_c = 1.7 \pm 0.1$, through a Monte Carlo procedure as before we get $M_v = (1.30^{+0.36}_{-0.29}) \times 10^{15} h^{-1} M_\odot$ at the 1σ confidence level for a mean X-ray temperature of 7 keV, where we have read the cluster number density from Fig. 1 of Bryan et al. (1994) to be $n(7 \text{ keV}, 0) = (1.6^{+1.6}_{-0.8}) \times 10^{-7} h^3 \text{ Mpc}^{-3} \text{ keV}^{-1}$. We can now use the calculated M_v to obtain the normalization that corresponds to the observed abundance of present-day galaxy clusters with mean X-ray temperature of 7 keV, which is given by equation (23). Again using $\delta_c = 1.7 \pm 0.1$ and a Monte Carlo procedure, we obtain $\sigma_8 = 0.62^{+0.17}_{-0.14}$. The errors represent 95 per cent confidence limits, and hence both the central value and the size of the uncertainty are very similar to those that we got for a pure CDM model, where $\sigma_8 = 0.60^{+0.19}_{-0.15}$ at 95 per cent confidence. It is encouraging that two rather different calculations give such similar answers. We shall use the relative errors obtained for a pure CDM model as they are slightly more conservative, and model the shift in the central value due to a change in Ω_ν by a simple linear fit:

$$\sigma_8 = (0.60 + 0.2\Omega_\nu/3)^{+32 \text{ per cent}}_{-24 \text{ per cent}}, \quad (36)$$

where the uncertainty is 95 per cent confidence. This relation will hold well for the models in which we are interested (indeed, it would be satisfactory just to employ the CDM result and ignore the slight shift in central value brought on by the hot component).

3.5 Abundance of high-redshift objects

To constrain the present-day power spectrum on scales around $1h^{-1} \text{ Mpc}$ requires detailed numerical simulations as those scales are well into the non-linear regime. However, a convenient alternative exists in the abundance of objects at high redshifts, which can sample the spectrum on those scales while they were still in the quasi-linear regime. The constraints on the *linear* power spectrum can then be evolved to the present day. In this context it is vital to recall that, when a hot dark matter component is introduced, perturbations on these scales can have their growth affected, typically growing more slowly than in a CDM model which has the effect of making the constraints weaker than naïve expectations. It is common to use analytic treatments based on rather nebulously defined neutrino Jeans masses to make this correction. Although this is often fine (since the corrections are typically small), we shall instead use direct calculations of the transfer functions at the appropriate redshift.

The most important objects for our purpose are quasars

and damped Lyman alpha systems, and we shall place particular emphasis on the latter as they provide stronger constraints. Uncertainties as to the efficiency of quasar formation and the number of quasar generations mean that only a lower bound on the power spectrum can be obtained from them at present. Damped Lyman alpha systems, on the other hand, in principle also offer an upper limit (though to our knowledge one has never been quoted), and indeed the evolution of the amount of gas in such systems as a function of redshift may well imply significant constraints on star formation.

For each object type, it is important to be as conservative as possible in supplying limits; the standard strategy is to obtain a rigid constraint that all models are compelled to satisfy, rather than a number with an error bar which can be subjected to a statistical test.

3.5.1 Quasars

The type of power spectra we are considering flatten towards short scales, so that, when one studies short scales, the relative influence of perturbations from larger scales becomes more important. Thus, in accordance with the discussion in Section 2.3, we should expect shear to become more important, and therefore the relative time efficiency for the formation of bound objects to decrease, as one considers the formation of increasingly smaller objects. Bearing this in mind, one should then expect the virialized dark haloes associated with the formation of galaxies to assemble more slowly than those for clusters due to the presence of a relatively stronger shear field^{**}. Even if these suppositions turn out to be correct it is difficult to quantify precisely both the strength of the shear field for a given scale at a certain epoch and its relation with the value one should consider for δ_c . It is due to this limitation that we will use in our analysis the abundance of the most luminous quasars at a redshift of $z = 4$, when the density field at the scale associated with the virialized dark haloes in which this type of quasar is embedded, which we will assume to have masses in excess of $10^{12} h^{-1} M_\odot$, is still not well developed and consequently shear can be ignored to a good approximation. We can therefore assume that these dark haloes collapsed nearly spherically and accordingly use the δ_c associated with spherical collapse. We will also assume that the time lag between halo virialization and quasar ignition is negligible. Adopting the most conservative result given by Haehnelt (1993) as corresponding to 95 per cent confidence, we have

$$\sigma(M = 10^{12} h^{-1} M_\odot, z = 4) \geq 0.26, \quad (37)$$

for an assumed quasar number density of around $5 \times 10^{-8} h^3 M_\odot \text{ Mpc}^{-3}$. The corresponding comoving scale is $R = 0.95 h^{-1} \text{ Mpc}$. However, this constraint is always weaker than that coming from damped Lyman alpha systems.

^{**} Some studies (Antonuccio-Delogu & Colafrancesco 1994) suggest that the presence of substructure within a collapsing object can increase its time-scale of collapse through dynamical friction. Though this effect, similarly to shear, delays collapse, its dependence on the shape of the power spectrum is the opposite, thus effectively diminishing the overall dependence of the time-scale of collapse on it. However, this effect seems not to be nearly as important as shear (Monaco 1995).

3.5.2 Damped Lyman alpha systems

At low and intermediate redshifts, $z \leq 2$, the most popular view is that the vast majority of the damped Lyman alpha lines which appear in the spectra of quasars are produced by neutral hydrogen present in quiescent large-scale discs, similar to those presently found in spiral galaxies. However, these disc systems would have to be 2 to 3 times bigger in size than present spiral galaxies in order to explain the apparent increase in filling factor with redshift, if one assumes that the comoving number density of these systems remains constant (Lanzetta, Wolfe & Turnshek 1995). An alternative explanation would be that this increase in filling factor with redshift is instead due at least partially to an increase in the comoving number density of disc systems with redshift, in particular for $1 < z < 2$. The excess number of systems would then disappear by merging, possibly giving rise to some of the presently observed elliptical galaxies. At higher redshifts, $z \geq 2$, there are some hints that these lines may be produced in objects more akin to turbulent protospheroids, from the apparent short time-scales of consumption of the neutral gas by star formation and the discrepancy between the observed low metallicities associated with the lines at those redshifts and the expected higher metallicity of the gas if it is to be the material from which disc stars in present spiral galaxies formed (Lanzetta et al. 1995). These protospheroids are the natural progenitors of galaxies, and the indication would then be that the transition between turbulent collapsing haloes and quiescent rotationally supported discs occurred at $z \sim 2$.

Instead of the widely quoted data of Lanzetta et al. (1995), we use the more recent data of Storrie-Lombardi et al. (1995) which revise downwards^{††} the estimated abundances at a redshift of around 3 and provide a new estimate at redshift 4. The strongest constraint comes from the redshift 4 data, though it is not significantly weakened if the redshift 3 data are used instead. We will present constraints from both.

Following the discussion in the previous subsection, we are interested in the amount of matter associated with damped Lyman alpha systems at redshifts 3 and 4. As we have seen, at these redshifts the systems are probably collapsing protospheroids massive enough to give rise to rotationally supported gaseous discs. The minimum total mass needed in order for that to happen seems to be around $10^{10} h^{-1} M_{\odot}$ (Haehnelt 1995), which corresponds to a circular velocity of 77 km s^{-1} . It is not clear how far these systems have collapsed gravitationally. A reasonable, and for our purposes conservative, hypothesis is that they have just collapsed along the first two collapsing axes, i.e. ‘filament’ formation, though the baryonic fraction of the collapsing material would have collapsed further through radiative cooling (e.g. Katz et al. 1994). Numerical studies indicate that a value of δ_c around 1.5 is associated with the time-scale of gravitational collapse along the first two collapsing axes (Monaco 1995), and accordingly we shall use it in the

Press–Schechter calculation. This gives a more conservative bound than $\delta_c = 1.7$.

In Storrie-Lombardi et al. (1995), the fraction of the critical density in the form of neutral gas associated with damped Lyman alpha systems at redshifts 3 and 4 is observed to be

$$\Omega_{\text{gas}}(z=3) = (0.0017 \pm 0.0003) h^{-1}, \quad (38)$$

and

$$\Omega_{\text{gas}}(z=4) = (0.0011 \pm 0.0002) h^{-1}. \quad (39)$$

The total amount of matter that was involved in the formation of damped Lyman alpha systems at these redshifts as a fraction of the critical density is then given by

$$\Omega_{\text{DLAS}}(z) = \frac{\Omega_{\text{gas}}(z)}{f_{\text{gas}} \Omega_{\text{B}}}, \quad (40)$$

where f_{gas} is the neutral fraction of the gas in those systems, which conservatively we will assume to be 1, and $\Omega_{\text{B}} = 0.016 h^{-2}$ is the cosmological baryon density given by standard nucleosynthesis. We now have to be careful in deciding which is the characteristic comoving mass scale associated with these systems. If one takes $10^{10} h^{-1} M_{\odot}$ to be the minimum mass of damped Lyman alpha systems then, because we do not expect massive neutrinos within the mass range we are considering to be able to cluster on this mass scale at $z \geq 2$, the characteristic comoving mass scale involved in the formation of these systems is given by $M = 10^{10} (1 - \Omega_{\nu})^{-1} h^{-1} M_{\odot}$. That is, it is originally perturbations on this larger mass scale that begin to collapse, but at some point during the collapse of the perturbations there will be a segregation between the massive neutrinos and the cold dark matter, the former remaining in an oscillatory mode roughly at the scale of segregation (approximately equal to the neutrino Jeans scale) and the latter collapsing further, eventually leading to the formation of $10^{10} h^{-1} M_{\odot}$ virialized objects.

All this therefore implies that the fraction $f(> M, z)$ of the total mass that is involved in the formation of damped Lyman alpha systems at redshifts 3 and 4 is given by

$$f(> M, z=3) > (0.106 \pm 0.033) h, \quad (41)$$

and

$$f(> M, z=4) > (0.069 \pm 0.021) h, \quad (42)$$

where $M = 10^{10} (1 - \Omega_{\nu})^{-1} h^{-1} M_{\odot}$. A 25 per cent uncertainty in the baryon fraction, corresponding loosely to 1σ , has been added in quadrature to the observational uncertainty. Since we want a lower bound on the density perturbation we take the 2σ lower end of the error bar. Using equation (12), we then have to a good approximation the 95 per cent confidence limits

$$\sigma(R, z=3) > 0.54 + 0.2h; \quad (43)$$

$$\sigma(R, z=4) > 0.50 + 0.2h, \quad (44)$$

for $0.3 < h < 0.7$, where $R = 0.2 (1 - \Omega_{\nu})^{-1/3} h^{-1} \text{ Mpc}$. In fact, the constraint is quite insensitive to the confidence limit chosen. We shall use the redshift 4 point as it provides the stronger constraint; although numerically the constraint is similar, it applies at a higher redshift.

^{††} Note that this still ignores the effect of gravitational lensing, which it is claimed can reduce the estimated abundance by a further 50 per cent (Bartelmann & Loeb 1996).

3.6 Compilation

Fig. 2 shows all the data we have discussed, plotted at the present epoch. The data on short scales, which are obtained at moderate redshift, are scaled to the present epoch assuming a pure CDM model (though in later analysis we shall directly apply the high-redshift transfer function). The *COBE* point is represented schematically as discussed. We have plotted the Peacock & Dodds points assuming a bias parameter (for *IRAS* galaxies) of 1.1, which is the best fit for $\Omega_0 = 1$; the errors shown on the individual points correspond to the errors on their relative location, and the uncertainty in bias, ± 0.2 (not illustrated in this Figure), then allows the entire data set to be shifted up or down.

This figure shows that the data follow a more or less continuous path, across a range of roughly four orders of magnitude both in linear scale and in the size of the dispersion. However, this large range makes the individual error bars very small, and were one to attempt to plot theoretical predictions on this figure it would be very hard to discern which were the best fit to the data.

In order to overcome this, we can use the knowledge that the standard CDM model, while unable to fit the observational data in detail, is certainly able to fit all of them to within a factor two or so. Consequently, we can greatly improve the graphical representation by plotting the observational data divided by the prediction of the *COBE* normalized standard CDM model. The choice of this particular model as the fiducial one is governed by history; it does not indicate any preference for this model over any other but rather is simply a graphical convenience. The data normalized to the standard CDM model are shown in Fig. 3.

As anticipated, the data all lie within a factor two or so of this canonical model, with the short-scale data falling below the prediction of *COBE*-normalized standard CDM. The Peacock & Dodds data (1994)^{††} are represented by a band, and the error bars on the end indicate the overall normalization uncertainty.

When we make comparisons of theory and observations, one of the aspects we have to take into account is that the data are available at different redshifts. When one has a hot dark matter component, the growth of perturbations on short scales is slower than in a CDM model, and this must be taken into account. Rather than impose an analytic approximation to the different growth rate, we directly use transfer functions calculated at the appropriate redshift of around $z = 3.5$.

It is fortunate that the data available at the present day are on scales large enough that the growth rate is the same as in the CDM model for these moderate redshifts, as seen in Fig. 1. This means that one can shift these data back to a redshift 3.5 in a model-independent way^{§§}. Consequently, the simplest approach is to consider all the data as given at redshift 3.5. Had we plotted this, it would look exactly as Fig. 2, but with the vertical axis divided by a factor 4.5 in ac-

cordance with the CDM growth law $\sigma(R) \propto 1/(1+z)$. Fig. 3 remains exactly the same, and when interpreted at this redshift one doesn't have to worry about perturbation growth suppression corrections in models with an HDM component.

Fig. 4 shows this data with some sample theoretical curves overlaid.

4 CONFRONTATION

Our three primary parameters are n , h and Ω_ν . Let us first specialize our discussion to varying single parameters of the standard CDM model. Although there is no clear motivation for adopting either $h = 0.5$ or $n = 1$ as standard values, this is the most common strategy in the literature.

4.1 Single-parameter variations

4.1.1 Scale-invariant CDM models

Since it was recognized that the CDM model could be fixed by lowering the shape parameter, considerable attention has been directed towards achieving this end by lowering the density parameter Ω_0 , usually retaining spatial flatness via the introduction of a cosmological constant. The alternative strategy to achieve this is to lower the Hubble parameter. With a slightly different motivation, such a strategy has been long advocated by Shanks (e.g. 1985). It was mentioned by Liddle & Lyth (1993a) and proposed as a possibility more vigorously by Bartlett et al. (1995). However, neither of those papers took advantage of the effect of the baryon content in these models, which can play a significant role in reducing the shape parameter, as given by equation (10). Consequently, it seems that the proposed $h \simeq 0.3$ may be too strict and we find that one can get away with $h \lesssim 0.35$. This is still a long way from the values currently discussed via direct observation (Freedman et al. 1994; Schmidt et al. 1994). However, the preferred baryon density may prove yet higher than the value we have adopted, say at the top of or beyond the range given by recent analyses of nucleosynthesis (Copi et al. 1995a,b), which would help to alleviate worries about the high baryon abundance in clusters if Ω_0 does turn out to be 1 (White et al. 1993b; White & Fabian 1995). Then high baryon density may become an increasingly attractive solution to the problems of standard CDM.

We remind the reader in passing that altering the number of massless species provides a way of mimicking the low- h power spectrum (Dodelson et al. 1994; White et al. 1995a) while retaining a higher true value of h .

4.1.2 Scale-invariant CHDM models

The idea of introducing a hot dark matter component to reduce the short-scale power relative to CDM has a long history (Shafi & Stecker 1984; Bonometto & Valdarnini 1984; Fang et al. 1984; Valdarnini & Bonometto 1985; Holtzman 1989; Schaefer et al. 1989; van Dalen & Schaefer 1992) and it quickly received a lot of attention (Schaefer & Shafi 1992, 1993; Davis, Summers & Schlegel 1992; Taylor & Rowan-Robinson 1992; Holtzman & Primack 1993) after the *COBE* observations. The most widely explored versions of the CHDM model assume both $n = 1$ and $h = 0.5$. As far as

^{††} We have left out from this figure the two points corresponding to the largest scales in order to obtain a clearer picture of the observations as these points are very close to the POTENT point.

^{§§} This is true only for CHDM models, and would not hold for open or cosmological constant models.

detailed simulation is concerned, the bulk of attention has gone to the choice $\Omega_\nu = 0.3$ (Davis et al. 1992; Klypin et al. 1994; Jing et al. 1994; Nolthenius, Klypin & Primack 1994; Yepes et al. 1994; Bryan et al. 1994; Klypin, Nolthenius & Primack 1995b).

The alternative approach, as adopted in this paper, is to investigate the parameter space more widely by concentrating on linear theory, and this has been done in many recent papers (Taylor & Rowan-Robinson 1992; Liddle & Lyth 1993b; Pogosyan & Starobinsky 1993, 1995a; Schaefer & Shafi 1994). Our aim here is to examine the widest possible parameter space using the most up-to-date linear theory constraints.

Should the recently claimed detections of the muon neutrino mass be confirmed, then it corresponds to a particular region of our parameter space. Assuming the standard abundance calculation, the possible LSND detection (Caldwell 1994; Primack et al. 1995) corresponds, with considerable uncertainty, to $\Omega_\nu = 0.1(h/0.5)^{-2}$; if h becomes smaller this corresponds to a greater fraction of the total density.

A much advertized drawback of the $\Omega_\nu = 0.3$ CHDM model is the possibility that it may not reproduce the observed abundance of damped Lyman alpha systems (Mo & Miralda-Escudé 1994; Kauffmann & Charlot 1994; Ma & Bertschinger 1994). This has led some to favour the reduction of Ω_ν to 0.25 or 0.20 (Klypin et al. 1995a). We find that, were we to make the same assumptions as they do concerning the *COBE* normalization and the damped Lyman alpha system abundance, we would more or less reproduce the constraint of Klypin et al. (1995a). Since their calculation is considerably more sophisticated than ours, being simulation based, one could regard this as a calibration of our calculation, though we have not had to do any tuning. Anyway, since their calculation was made, things have generally gone in the direction of weakening the early galaxy formation constraint on the CHDM model; the *COBE* normalization has gone up slightly, and more recent damped Lyman alpha system abundance observations (Storrie-Lombardi et al. 1995) have produced lower results than those of Lanzetta et al. (1995). Consequently, we find that the constraint from damped Lyman alpha systems on *COBE*-normalized CHDM models has weakened somewhat, back to $\Omega_\nu \lesssim 0.30$ for the $n = 1$, $h = 0.5$ version.

However, a more important question is what values of Ω_ν are preferred when one brings other data into play. Fig. 4 shows that there are already indications from the shape of the galaxy correlation function that the $\Omega_\nu = 0.30$ model is subtracting too much short-scale power. A better eyeball fit to the correlation function data from Fig. 4 is $\Omega_\nu = 0.20$. This sort of value has been criticized on alternative grounds, that it may overproduce clusters (Primack et al. 1995), a problem made worse by the higher normalization but slightly improved by our view that the cluster constraint is weaker than usually advertized. Fig. 5 illustrates our allowed parameter region as a function of both Ω_ν and h . The combination of cluster and damped Lyman alpha system abundance appears sufficient to exclude all models with $h \geq 0.55$. For values of $\Omega_\nu \lesssim 0.3$ the cluster constraint alone is what limits the value of h . [The model predictions of the cluster amplitude would, however, be compatible with the constraint for higher h if Ω_ν were composed of more than one degenerate mass flavour (Primack et al. 1995; Babu et al. 1996).]

Fig. 5 makes it clear that there is also a lot of extra freedom to be gained via fairly modest decreases in h , with a wide band of allowed values opening up. If the LSND detection corresponds to a single neutrino, giving $\Omega_\nu \sim 0.1(h/0.5)^{-2}$, it cuts across the allowed region around $h = 0.4$, with considerable uncertainty.

Overall, our analysis suggests that, largely due to the higher *COBE* normalization, the parameter space of scale-invariant CHDM is not too large. However, we shall see that, when one allows n to vary and includes the possibility of gravitational waves, the freedom becomes greater.

4.2 Tilted CDM models

Let us now extend the discussion to take in the general class of inflation-based cold dark matter models. Naturally, one chooses Ω_ν to be zero, but the parameters n and h are to be freely varied and gravitational waves added if desired.

In the context of an arbitrary choice for the initial perturbation spectrum, the possibility of choosing a spectral index other than $n = 1$, which has now become known as ‘tilt’, has often been discussed. The modern context, where the origin of the tilt is identified as inflation and a connection made to the desired slope of the galaxy correlation function, was discussed by Bond (1992), Liddle, Lyth & Sutherland (1992), Cen et al. (1992), Adams et al. (1993) and Liddle & Lyth (1993a). After the original *COBE* result came out (Smoot et al. 1992) the prognosis for such models was not particularly good: the necessary tilt to explain the galaxy correlation function, especially as witnessed by the APM survey (Maddox et al. 1990), left a perceived deficit of short-scale power when adjusted to the *COBE* data. Since then the situation has improved somewhat, due to the higher normalization of the current *COBE* data (Górski et al. 1996). It appears from Fig. 4 that, for $h = 0.5$, even tilting to $n = 0.7$, a commonly discussed number, is not sufficient to get the slope of the galaxy correlation function right, and one has to go even lower. Substantial gravitational waves, as there would be in a power-law inflation model, would make things yet worse, so for values of h near 0.5 the implementation must be in a model such as natural inflation (Adams et al. 1993) which predicts negligible gravitational waves.

Fig. 6 shows contour plots of the constraining observations in the n - h plane. The top panel is with no gravitational waves; the lower panel adopts the power-law inflation amplitude of gravitational waves for $n < 1$ and zero otherwise, as discussed in Section 2.3. This figure confirms the viability of scale-invariant CDM provided that h is low enough. More importantly, it shows that there may still be a reasonable amount of parameter space available for CDM models. Provided that n is lowered sufficiently, these can work for values of h up to about 0.5, but not for any higher values. The parameter space widens out to low values of h , so no useful lower limit on h can be obtained this way. The incorporation of gravitational waves reduces the favoured area.

Although we have fixed the baryon density, in the regime where the spectrum is well described via the shape parameter defined by equation (10) one can account for a variation by defining an effective h value. For the range of Ω_B allowed by nucleosynthesis the change is always small, and via a small parameter expansion one can write $h_{\text{eff}} \simeq h(1 - 2\Delta\Omega_B)$, where $\Delta\Omega_B = \Omega_B - \Omega_B^{\text{unc}}$. This allows one to

interpret a point in Fig. 5 as representing a range of models with slight simultaneous variation in Ω_B and h satisfying this relation. However, within the nucleosynthesis range only small changes can be made.

White et al. (1995b) analysed two particular versions of the CDM model. In their favoured models, the desired reduction in short-scale power is brought about by accumulating small changes from all sources. They considered a slightly higher baryon density which can be incorporated as just discussed. Without gravitational waves, they favoured $n = 0.8$ and $h = 0.45$ (their higher Ω_B giving $h_{\text{eff}} = 0.43$), which is at the edge of our favoured parameter range. However, with gravitational waves they preferred $n = 0.9$, still with $h = 0.45$, which our results do not favour; our treatment of the shape of the galaxy correlation function is more stringent than theirs and a smaller value of h would be required.

4.3 Tilted CHDM models

The full parameter space is best investigated by running slices through the volume of n - h - Ω_ν space. Following Pogosyan & Starobinsky (1995a), we show cuts of constant n and of constant h . We do each of these for both the case without gravitational waves and the case with power-law inflation gravitational waves.

Figs 7 and 8 show the slicings for the case of no gravitational waves. Fig. 7 includes a reproduction of the scale-invariant CHDM case shown in Fig. 5. As anticipated from all the special cases we have already examined, there is a fairly reasonable parameter space available which explains all the observational data. Unless h is below 0.5, a component of HDM appears to be required. Larger values of h , up to around 0.65 in the most extreme cases, are then permitted provided that one introduces a strong tilt as well the hot component.

As regards Ω_ν , it seems that the highest it can reach is 0.35 in the rather extreme case of $h = 0.4$ and $n \simeq 1.2$. For n , the lowest working value is $n = 0.6$, while concerning high values n above 1.2 is possible provided that a very low value of h is tolerated. In this regard, our conclusions favour those of Pogosyan & Starobinsky (1995a) rather than Lucchin et al. (1995) in that we see no particular advantage in adopting a blue ($n > 1$) spectrum and find that values above 1.1 can be maintained only via a dubiously low Hubble parameter. Borgani et al. (1995) have suggested that the adoption of a blue spectrum helps to alleviate worries about damped Lyman alpha system abundance; while in isolation this is certainly true we find that this strategy is not favoured by other data^{¶¶}.

Figs 9 and 10 show equivalent slicings for the case where power-law inflation gravitational waves are included for $n < 1$. For $n \geq 1$ they are the same as Figs. 7 and 8. They show that n has to be above 0.8 before any sort of fit to the data is available. This is part of a fairly general result that the incorporation of gravitational waves reduces

the total available parameter space. Only a few narrow slivers of extra parameter space are opened up by the inclusion of gravitational waves. This is because of the strength of the shape parameter constraint, which forces a fairly dramatic reduction in short-scale power. Models able to fit this are typically not able to lose much more of this power by permitting some of the *COBE* signal to be soaked up by gravitational waves.

5 DISCUSSION

Despite the recent attention directed at low-density models of structure formation, either open or with a cosmological constant, the possibility of working models with the critical density remains an attractive one. We have been able to show that the current observational constraints continue to allow a substantial amount of parameter space for these models.

An important sub-class of the models that we have discussed is generalized CDM models. Here all the dark matter is assumed to be cold, and one attempts to fit the data by permitting variation of the Hubble parameter and the initial conditions (tilt and gravitational waves) coming from inflation. We illustrated the constraints in Fig. 6; they demonstrate that there is still an area of parameter space available for CDM models. The most plausible situation requires a tilt to $n < 1$ and not too high a gravitational wave amplitude. The biggest drawback these models face is that they require a low value of the Hubble parameter; a strong tilt just permits $h = 0.5$, but that is the highest in any region of parameter space. The bulk of available parameter space, where the tilt is not so strong, requires h some way below this. This sits uncomfortably with recent direct measurements of the Hubble constant (Freedman et al. 1994; Schmidt et al. 1994), though we remind the reader again that power spectra mimicking low values of h can be achieved by introducing extra massless species/decaying particles.

In the case of the scale-invariant CHDM models, we find that they remain viable at $h = 0.5$, but hardly any higher. However, when one introduces the additional freedom of varying n , a much more substantial parameter region opens up which does allow fits at higher values of h . A value $h = 0.6$, consistent with recent direct measurements, seems easy to achieve provided that one introduces a strong enough tilt along with the HDM component, and perhaps even $h = 0.65$ is possible if one pushes right to the corner of the parameter space (though the age of the universe in such a model could be problematic).

We have not investigated the introduction of gravitational waves as thoroughly as the other parameters, but we have looked at the case where the amplitude is that given by power-law inflation. What we find is that the gravitational waves are not very helpful; they lead to a reduction in the allowed parameter ‘volume’ and only in very limited regions do they allow working models for values of n , h and Ω_ν that would fail without gravitational waves. However, that said, even with this rather large gravitational wave component there are still significant allowed regions. Large-scale structure is therefore not able to exclude the possibility of such gravitational waves.

To conclude, we have presented an extensive comparison

^{¶¶} After our paper was submitted, Borgani et al. (1995) was revised to include discussion of the cluster abundance, obtaining conclusions similar to ours.

of critical-density models for structure formation with linear and quasi-linear observational data. We have calculated new transfer functions and provided an improved fitting formula for them which gives the power spectra as a continuous function of all of n , h , Ω_ν , Ω_B and z . We have interpreted the data in terms of the dispersion of the density contrast smoothed on some scale R . We have found a substantial allowed parameter space for CDM and CHDM models, which at least in the latter case seems likely to survive for some time to come. Critical-density models continue to offer a viable and aesthetically simple basis for understanding structure formation.

ACKNOWLEDGMENTS

ARL is supported by the Royal Society, RKS and QS by DOE DE-FG02-91ER, NASA NAG5-2646 and the Bartol Research Institute, and PTPV by the PRAXIS XXI programme of JNICT (Portugal). ARL and DHL thank the Aspen Center for Physics, where discussions pertinent to this paper took place. We thank Stefano Borgani, Martin Hendry, Lev Kofman, Cedric Lacey, John Peacock, Dmitry Pogosyan, Joel Primack, Douglas Scott and Martin White for helpful discussions on a variety of topics. ARL and PTPV acknowledge the use of the Starlink computer system at the University of Sussex.

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FIGURE CAPTIONS

Figure 1: We plot the dispersion $\sigma(R, z)$ for two different CHDM models, $\Omega_\nu = 0.2$ and 0.3 , at redshifts of zero (solid lines) and 3.5 (dashed lines). The lower curves correspond to $\Omega_\nu = 0.3$. The curves have been normalized on to each other at large scales, which is achieved by using the CDM growth law $\sigma(R, z) \propto (1+z)^{-1}$. Except for the shortest scales, the transfer functions are redshift-independent, indicating that the CDM growth law holds for $R \geq 3h^{-1}$ Mpc.

Figure 2: The observational data that we consider, interpreted in terms of $\sigma(R)$. Error bars are 1σ and lower limits are 95 per cent confidence. We represent the *COBE* data schematically at $4000h^{-1}$ Mpc as discussed in text; they are indicated by a filled square the size of which roughly represents the uncertainty. The Peacock & Dodds data are shown by circles (as discussed, we omit the leftmost four points); there is an uncertainty in overall normalization which has not been illustrated. The bulk flow constraint is represented by a star, and the cluster abundance constraint by a cross. The lower limits on the left hand side correspond to damped Lyman alpha systems (leftmost, values for redshifts 3 and 4 overlap) and quasars (right). Although the data clearly show a smooth trend, they cover such a range in $\sigma(R)$ values that one cannot use a figure of this form to compare models by eye.

Figure 3: As Fig. 2, but plotting $\sigma(R)$ relative to its value in the *COBE*-normalized standard CDM model. This greatly improves clarity. The data points are as in Fig. 2 [*COBE*, filled square; bulk flows, star; cluster abundance, cross; damped Lyman alpha system (now shown at two different redshifts) and quasar abundances, lower limits] except that we now show the Peacock & Dodds data as a band representing the 1σ errors about the (unplotted) central values. The error bars on the end of the band indicate their estimate of the uncertainty in overall normalization of this data set. We see that the data are not well fitted by the standard CDM model, which possesses too much short-scale power. Although to a reasonable accuracy this figure applies at any epoch, it is most accurately applied at redshift 3.5 corresponding to the quasar and damped Lyman alpha system abundances, so that one need not worry about

the suppressed perturbation growth rate in models with an HDM component.

Figure 4: The data plotted as in Fig. 3, with some illustrative theoretical curves overlaid for comparison. These curves are those appropriate to redshift 3.5, as discussed in the text. We have shown only examples where a single parameter of the standard CDM scenario has been modified. The solid line is the standard CDM model; the others modify one parameter from this fiducial model, as indicated in the key. All models are precisely *COBE* normalized; the *COBE* point at $4000h^{-1}$ Mpc is illustrative. Remembering that one can shift the entire Peacock & Dodds data set vertically, reasonable eyeball fits to the data are possible via any of the following: lowering h to about 0.35, lowering n to about 0.7 assuming no gravitational waves, introducing a hot dark matter component at about the $\Omega_\nu = 0.2$ level.

Figure 5: Scale-invariant CHDM models. The lines shown are from the galaxy correlation data (dotted), cluster abundance (dashed) and damped Lyman alpha systems (solid). Shading indicates the favoured area. All constraints are plotted at 95 per cent confidence.

Figure 6: Contour plots of constraining observations for CDM models. The upper panel is without gravitational waves; the lower panel includes power-law inflation gravitational waves for $n < 1$. The lines shown are galaxy correlations (dotted), cluster abundance (dashed), damped Lyman alpha system abundance (solid) and POTENT (dot-dashed). Shading indicates the favoured area and all data are plotted at 95 per cent confidence.

Figure 7: Four slices through the h - Ω_ν plane at different n , with no gravitational waves included. The values of n are as indicated, and the data are plotted as in Fig. 6.

Figure 8: Four slices through the n - Ω_ν plane at different h , with no gravitational waves included. The values of h are as indicated, and the data are plotted as in Fig. 6.

Figure 9: Four slices through the h - Ω_ν plane at different n , with gravitational waves included. The values of n are as indicated, and the data are plotted as in Fig. 6.

Figure 10: Four slices through the n - Ω_ν plane at different h , with gravitational waves included. The values of h are as indicated, and the data are plotted as in Fig. 6.